A Ray-driven Approach to Analytical SPECT Reconstruction of Non-uniform Attenuation with Variable Focal-length Fan-beam Collimators

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Abstract-- **Single-photon emission computed tomography (SPECT) is based on the measurement of radiation emitted by a radiotracer injected into the patient. Because of photoelectric absorption and Compton scatter, the γ photons are attenuated inside the body before arriving at the detector. A quantitative reconstruction must consider the attenuation, which is usually non-uniform. Novikov had derived an explicit formula for non-uniformly attenuated SPECT reconstruction of parallel-beam collimators. In this paper, we extend their researches to variable focal-length fan-beam collimators, and derived a ray-driven analytical variable fan-beam reconstruction formula with non-uniform attenuation. Its accuracy is demonstrated by computer simulation experiments*.**

I. Introduction

Single-photon emission computed tomography (SPECT) reconstructs the image of a radiotracer or radiopharmaceutical uptake distribution within the body from the measurement of decayed γ ray radiation, where the radiotracer is injected intravenously into the patient and the uptake image at a region of interest (ROI) reflects directly the cell function there. Because of photoelectric absorption and Compton scatter, the γ photons are attenuated inside the body before arriving at the detector. Tretiak and Metz [4] developed an explicit inversion formula of uniformly attenuated (or exponential) Radon transform in two dimensions for parallel-beam collimator geometry. This algorithm assumes that the attenuation coefficient is a constant across the body and that the body contour is convex. Some alternative inversion algorithms were developed later with extension to more complicated collimator geometries. Weng, *et al.* [7] extended the parallel-beam algorithm to fan-beam collimator geometry by a coordinate transform. You, *et al.* [8] derived a Cormack-type inversion of the exponential Radon transform by employing the circular harmonic transform directly in both the projection space and the image space, instead of the Fourier space. It was applied to parallel-beam, fan-beam and variable focal-length fan-beam collimator geometries. However, all of these algorithms are limited to uniform attenuation, which is not applicable for imaging of non-uniform attenuation, such as cardiac studies, where a quantitative reconstruction must consider the non-uniform attenuation among the lungs, soft tissues and rib bones.

Novikov [2] had derived an explicit inverse formula for non-uniformly attenuated Radon transform of parallel-beam collimator geometry in a framework within the content of conventional filtered backprojection (FBP), although other attempt had been reported [3]. This explicit formula had been implemented and good reconstruction results were obtained [1]. For clinical applications, however, fan-beam and variable focal-length fan-beam collimators would be preferred for brain and chest imaging. Fan-beam collimator is used to improve the sensitivity and resolution for imaging small organs like the brain. Variable focal-length fan-beam collimator overcomes the truncation problem of fan-beam collimator when a small organ is embedded in a large body, such as the heart in the chest. We have extended the previous researches on parallel-beam geometry to fan-beam and variable focal-length fan-beam collimators by an efficient approximated algorithm [5], where good reconstruction results were reported. Because any ray in non-parallel beam geometry can be treated as a ray in parallel-beam geometry, then we derived an accurate algorithm for fan-beam geometry [6], where improvement in reconstruction was observed. In this work, we extended the accurate algorithm to variable focal-length fan-beam geometry.

II. Basic Notations

The notation of [2] will be used throughout this paper. Let $f(x)$ denote the object function to be reconstructed and $a(x)$ be the attenuation coefficient across the body. Let $g_\varphi(p)$ be the projection datum at position p with projection angle φ.

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In order to simplify the derivation, a rotated coordinate system (s,p) is introduced, see Figure 1,

$$s = x\cos\varphi + y\sin\varphi \\ p = -x\sin\varphi + y\cos\varphi \quad (1)$$

then the projection has the form of

$$g_\varphi(p) = \int_{-\infty}^{\infty} \exp(-D_\varphi a_\varphi(s,p)) f_\varphi(s,p)ds \quad (2)$$

where the divergent beam transform is expressed as

$$D_\varphi a_\varphi(s,p) = \int_s^{\infty} a_\varphi(s,p)ds \quad (3)$$

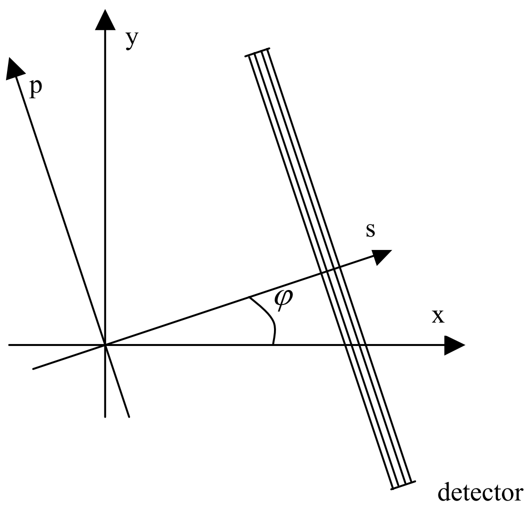

Fig. 1: The rotated coordinate system.

III. Reconstruction for Parallel-beam Collimator

According to [1, 2, 3], parallel-beam reconstruction formula can be written as follows,

$$f(x) = \frac{1}{4\pi}\int_0^{2\pi} \frac{\partial}{\partial p}(e^{(D_\varphi a_\varphi)(s,p)} ga_\varphi(p))d\varphi \quad (4)$$

where

$$ga_\varphi(p) = e^{-A_\varphi(p)}[\cos(B_\varphi(p))H\left(\cos(B_\varphi(p))e^{A_\varphi(p)}g_\varphi(p)\right) + \sin(B_\varphi(p))H\left(\sin(B_\varphi(p))e^{A_\varphi(p)}g_\varphi(p)\right)]$$

$$A_\varphi(p) = \frac{1}{2}Ra_\varphi(p)$$, and

$$B_\varphi(p) = HA_\varphi(p) \quad (5)$$

where H denotes the Hilbert transform [2] and R represents the Radon transform [2].

Formula (4) can also be replaced by the following formulas:

$$f(x,y) = \frac{1}{4\pi}(\frac{\partial}{\partial x}B_C(x,y) + \frac{\partial}{\partial y}B_S(x,y)) \quad (6)$$

$$B_C(x,y) = \int_0^{2\pi}[e^{(D_\varphi a_\varphi)(s,p)} ga_\varphi(p)]\cos\varphi\, d\varphi$$

$$B_S(x,y) = \int_0^{2\pi}[e^{(D_\varphi a_\varphi)(s,p)} ga_\varphi(p)]\sin\varphi\, d\varphi \quad (7)$$

So, the parallel-beam reconstruction process can be described as follows

a) Computing the divergent beam transform $D_\varphi a_\varphi(s,p)$ and the Radon transform $Ra_\varphi(p)$.

b) Computing the modified projections $ga_\varphi(p)$.

c) Differentiating the result of $e^{(D_\varphi a_\varphi)(s,p)} ga_\varphi(p)$ in p.

d) Performing back-projection.

Or:

a) Computing the divergent beam transform $D_\varphi a_\varphi(s,p)$ and the Radon transform $Ra_\varphi(p)$.

b) Computing the modified projections $ga_\varphi(p)$.

c) Computing the vector $\vec{\varphi} e^{(D_\varphi a_\varphi)(s,p)} ga_\varphi(p)$ for each point and the sum of these vectors in deferent angles φ, which is from 0 to 2π as shown by formula (7).

d) Computing the divergence of formula (6).

IV. Reconstruction for Variable Focal-length Fan-beam Collimator

In variable focal-length fan-beam collimator system, the focal length varies according to the distance from this collimator to the center of the system. Figure 2 shows the variable focal-length fan-beam collimator system.

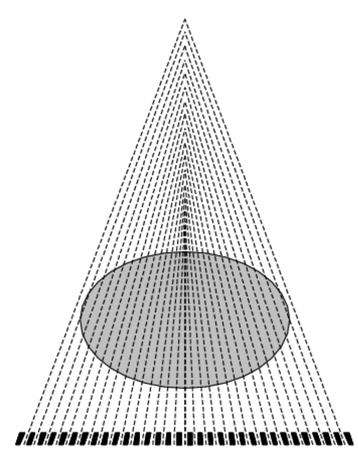

Fig. 2: Variable focal-length fan-beam collimator system.

Any ray (p,β) in variable focal-length fan-beam geometry can be seen as a ray (x_r,θ) in parallel-beam geometry. Let $D(p)$ be the variable focal length. The relation between the parallel and non-parallel geometries is:

$$\theta = \beta + \gamma = \beta + tg^{-1}\frac{p}{D(p)}$$
$$x_r = p\cos\gamma = \frac{pD(p)}{\sqrt{D(p)^2 + p^2}} \qquad (8)$$

For each ray (p, β), we can build a local coordinate system $(x^{"}, y^{"})$. The relation between this local coordinate system and the original coordinate system is:

$$x^{"} = x\cos\theta + y\sin\theta$$
$$y^{"} = -x\sin\theta + y\cos\theta \qquad (9)$$

In this local coordinate system (see Figure 3), this ray will be a parallel-beam ray. So we can use the parallel-beam formula to process this ray. For any point (x, y), its position in this local coordinate system is $(x^{"}, y^{"})$. A ray-driven analytical fan-beam reconstruction formula can be written as follows

$$f(x,y) = \frac{1}{4\pi}\left(\frac{\partial}{\partial x}B_C(x,y) + \frac{\partial}{\partial y}B_S(x,y)\right)$$

$$B_C(x,y) = \int_0^{2\pi}\int_{-\infty}^{\infty}[e^{(D_\theta a_\theta)(x^{"},y^{"})} ga_{(\theta,x_r)}(x^{"})]\cos\theta dx_r\, d\theta$$
$$B_S(x,y) = \int_0^{2\pi}\int_{-\infty}^{\infty}[e^{(D_\theta a_\theta)(x^{"},y^{"})} ga_{(\theta,x_r)}(x^{"})]\sin\theta dx_r\, d\theta \qquad (10)$$

where

$$ga_{(\theta,x_r)}(x^{"}) =$$
$$e^{-A_\theta(x^{"})}[\cos(B_\theta(x^{"}))H\left(\cos(B_\theta(x_r))e^{A_\theta(x_r)}g(p,\beta)\delta(x^{"} - x_r)\right)$$
$$+\sin(B_\theta(x^{"}))H\left(\sin(B_\theta(x_r))e^{A_\theta(x_r)}g(p,\beta)\delta(x^{"} - x_r)\right)]$$

and

$$B_\theta(x^{"}) = HA_\theta(x^{"}),\ A_\theta(x^{"}) = \frac{1}{2}Ra_\theta(x^{"}) \qquad (11)$$

where H and R have been defined before. The relation between (x, y) and $(x^{"}, y^{"})$ is shown by equation (9). The relation between (p, β) and (x_r, θ) is shown by equation (8).

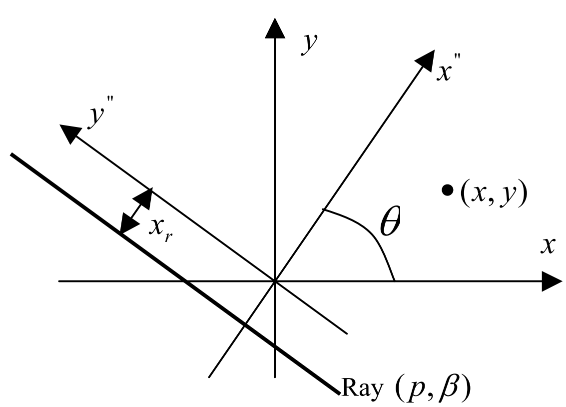

Fig. 3: The local coordinate system.

V. SIMULATIONS

An experimental study was carried out to test the derived reconstruction formulas using different mathematical phantoms with non-uniform attenuation on an image array of 128x128 size (see top left and middle of Figure 4). By setting the attenuation coefficients to be zero, our reconstruction formula generated the result on the top right of Figure 4, which is identical to the FBP-type reconstruction [9], where the strong effect of non-uniform attenuation is seen (dark and deformation in the middle two circles, and artifacts in the reconstructed image). Setting the focal length as to be ∞, which means the parallel-beam geometry, our reconstruction result on the bottom left of Figure 4 is the same as that reconstructed by the non-uniform attenuation parallel-beam reconstruction formula [1, 2]. With a variable focal-length fan-beam collimator of $D(p)$ = 200+10$|p|$, we obtained almost identical result as the original image, see bottom middle of Figure 4. By another variable focal-length fan-beam collimator with variable focal-length function of $D(p)$ = 300+30$|p|$, the same result was obtained, showing its robustness for different focal-length collimators. To see its performance on different objects, we tested our algorithm on another mathematic Shepp-Logan phantom. The comparison study between the parallel-beam geometry and the variable focal-length fan-beam collimator of $D(p)$ = 300+30$|p|$ is shown by Figure 5.

VI. MULTIPLICATIVE CORRECTION METHOD

There were some artifacts in the reconstructed images. These artifacts were also reported in [1] for parallel-beam geometry. These artifacts may be due to the cause of the interpolation error. There are ways to correct them. The multiplicative correction method proposed by Kunyansky [1] is one choice to eliminate these artifacts, which has shown noticeable improvement. This method is described as follows:

1, Using the basic algorithm to obtain the approximate reconstruction image $f^{appr}(x, y)$.

2, Computing $R_0 f^{appr}$ (by setting the attenuation coefficient to be 0) and $R_\alpha f^{appr}$.

3, Calculating the ratio

$$K_\varphi(p) = \begin{cases} R_{0,\varphi}f(p)/R_{a,\varphi}f(p), & R_{a,\varphi}f(p) > 0 \\ 0, & R_{a,\varphi}f(p) = 0 \end{cases}.$$

4, Multiplying $g_\varphi(p)$ by $K_\varphi(p)$.

5, Reconstruct image using the basic method with attenuation coefficient set to 0, or FBP method.

Figure 6 shows the results before (on the left) and after (on the right) using the multiplicative correction method with our variable focal-length fan-beam collimator algorithm. Although this correction method can eliminate most of the artifacts, it is not the best one to eliminate all the artifacts. It may generate some new artifacts during the processing. Alternative methods that can eliminate the artifacts are in progress.

VII. DISCUSSIONS AND CONCLUSION

From Fig. 4 and Fig.5, we can see that there are some artifacts in the reconstruction image, and the artifacts in variable fan-beam reconstruction image are more than those in the parallel-beam reconstruction image. These artifacts are the cause of the error of interpolation. There are more non-parallel rays in the variable focal-length fan-beam collimator geometry than those in parallel-beam collimator geometry, which lead to more interpolation error and more artifacts in variable focal-length fan-beam collimator geometry.

Our method is ray driven. It computes the results by a ray-by-ray manner. This may cost more computing time, as compared to the parallel-beam reconstruction algorithm. But this method is an exact analytical variable fan-beam reconstruction formula. The experiments demonstrated the accuracy of the derived formulas for analytical reconstruction of non-uniformly attenuated SPECT with variable focal-length fan-beam collimators.

VIII. REFERENCES

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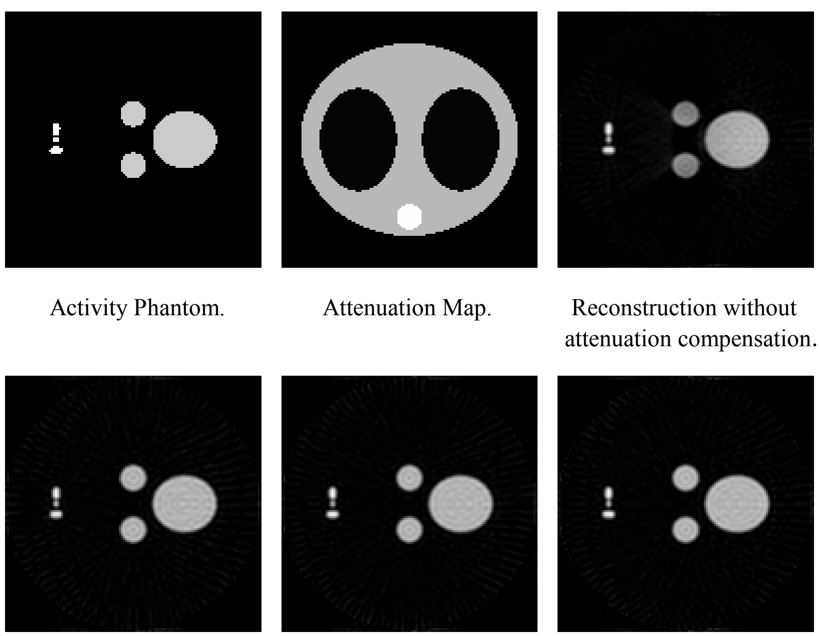
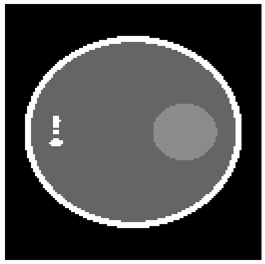


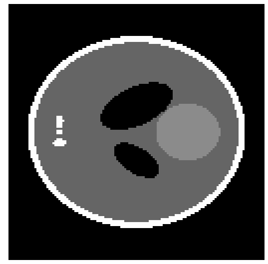
Activity Phantom. Attenuation Map. Reconstruction without attenuation compensation.

Parallel-beam $D(p) = \infty$. $D(p) = 200+10|p|$. $D(p) = 300+30|p|$.

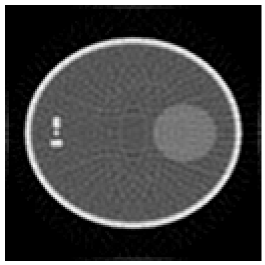
Fig. 4: Reconstruction results of our method for different conditions.
($D(p)$ is focal length function in pixel units, and image size is 128×128).



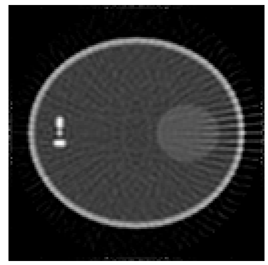
Activity Phantom.



Attenuation Map.

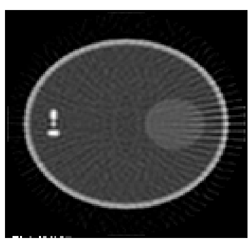


Parallel-beam $D(p) = \infty$.

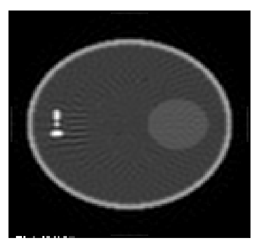


$D(p) = 300{+}30|\,p\,|$

Fig. 5: Reconstruction results #2 using our method.
($D(p)$ is focus length function, unit is pixel, image size is 128×128).



Variable fan beam
$D(p) = 300{+}30|\,p\,|$.



Variable fan beam $D(p) = 300{+}30|\,p\,|$
with multiplicative correction.

Fig. 6: Multiplicative correction method.
($D(p)$ is focal length function in pixel units, and image size is 128×128).