An Exact Modeling of Signal Statistics in Energy-integrating X-ray Computed Tomography

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ABSTRACT

Energy-integrating detection of x-ray sources is widely used by modern computed tomography (CT) scanners and has been an interesting research topic for the purpose of more accurately processing the data toward low-dose applications. While the energy-integrating detection can be described by a compound Poisson distribution, this work provides an alternative means to explicitly consider the Poisson statistics of the quanta and the energy spectrum of the x-ray generation. An exact solution for the first two orders of the compound Poisson statistics is presented. Given the energy spectrum of an x-ray source, the mean and variance of the measurement at any count-density level can be computed strictly. This solution can provide a quantitative measure on the condition under which an assumption of employing the most commonly-used independent identical distribution (i.i.d.), such as Gamma, Gaussian, etc, would be valid. A comparison study was performed to estimate the introduced errors of variance by using these substitute statistical functions to approximate the actual photon spectrum. The presented approach would further be incorporated in an adaptive noise treatment method for low-dose CT applications.

Keywords: Low-dose CT, energy-integrating detection, energy spectrum, compound Poisson statistics

1. INTRODUCTION

In x-ray computed tomography (CT), Poisson noise model has been widely used in noise simulation and image reconstruction studies which assume a quanta-counting process [1-4]. This model, however, would only be applicable for the photon counting detector systems used in, e.g., SPECT (single photon emission computed tomography) and PET (positron emission tomography) because the photon energy generated by an x-ray source is rather heterogeneous. The shape of the energy spectrum is the result of the alternating voltage applied to the x-ray tube, the multiple bremsstrahlung interactions between the accelerated electron and the target material and the filtration in the pathway of the beam. The energy spectrum consists of a continuous distribution of energies due to the bremsstrahlung effect and a discrete distribution due to the characteristic radiation. It is very difficult to characterize the beam quality in terms of energy, penetrating power, or degree of beam hardening. A roughly approximation of one-third of the maximum energy (kVp) was often used to state the average or effective energy in photon-counting approaches. This approximation may be acceptable for the estimate of the first order statistics of the energy-integrating detection and may not be valid for higher orders, which are usually needed for low-dose applications.

Modifying the Poisson model to include the heterogeneous nature of x-rays has been attempted by several studies, e.g., [5] for image reconstruction, [6-8] for sinogram restoration. From a different view point, Whiting, et al. [9, 10] derived a probability density function (PDF) to describe the statistics of the energy-integrating detection process, called compound Poisson (CP) statistics, where a underline assumption was made that the number of x-ray quanta within an energy interval in the spectrum follows the Poisson distribution and the summation of all energy intervals over the spectrum leads to the detected signal. Based on the derived compound Poisson PDF, a comparison study was performed for maximum likelihood (ML) image reconstruction between the compound Poisson model (where the derived PDF was

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used as the likelihood function) and the Poisson model [11]. The comparison study revealed the complexity in utilizing the derived PDF as the cost function for image reconstruction. Effort has been devoted to mitigate the complexity [12], where an exponential dispersion (ED) model was proposed with Gamma approximation to the compound Poisson statistics. The simplified model was applied as a cost function for ML image reconstruction algorithm. Although the above ML image reconstruction studies did not show noticeable difference between the compound Poisson and the Poisson descriptions of the energy-integrating detection process, further investigation of the compound Poisson statistics of energy-integrating detection is worth for both theoretical advancement and practical use under other cost functions.

In this work, we present an alternative, exact solution to the first two orders of the compound Poisson statistics on the energy-integrating detection of x-ray signal. Given the energy spectrum of a modern CT scanner, the mean and variance of the measurement at any count-density level can be computed strictly. It provides a quantitative measure on any approximation to the CP statistics and also a statistical model to be used for adaptive noise treatment.

In section II, we first investigate the physics of photon emission and detection for x-ray CT modality and recently proposed CP model which incorporates polyenergetic x-ray source. Based on the CP model, we further investigate the statistic moment and take real photon spectra into consideration.

2. METHODS

2.1 Physical model

X-ray radiation is produced whenever a substance is bombarded by high speed electrons. It can be generated by an x-ray tube, a vacuum tube that uses a high voltage to accelerate electrons released by a hot cathode to a high velocity. The high velocity electrons collide with a metal target, the anode, creating the x-rays.

The maximum energy of the produced x-ray photon is limited by the energy of the incident electron, which is equal to the voltage on the tube. The typical energy range of the x-ray photons generated for medical CT is between roughly 20 keV and 140keV. The measured spectra obtained from commercial CT scanners are evident that the spectra is consists of a continuous spectrum, see in figure. 1.

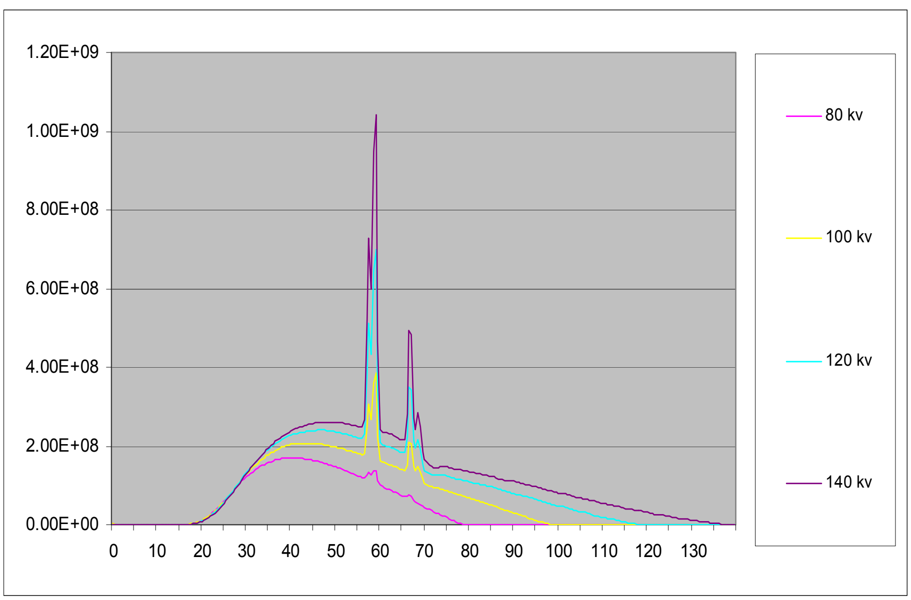

Fig.1: X-ray photon energy spectra for four different energy settings of 80, 100, 120, and 140 kVp.

The measured spectral distribution of the photons emitted from a diagnostic x-ray tube excited at 80, 100, 120 and 140 kV respectively. The figure gives the relative number of photons in each energy interval as a function of photon energy. The area under the curves is proportional to the total number of photons emitted. From these measured spectra it is evident that the spectrum consists of a continuous spectrum and discrete sharp spikes superimposed in it.

2.2 Compound Poisson model

Though the actual photon energy from medical CT consists of continuous spectrum, conventionally, the commonly used model for noise estimation of CT data considers a monoenergetic x-ray source that generates quanta attenuated by the scanned object and collected by detectors which count the number of survived quanta. The signal generated thus follows Poisson statistics. Though it explains general observed trends of signals with noise exposure and object attenuation, it is not able to reflect the physical characteristics of actual modality.

Analysis over the detected x-ray spectra directly that aimed to obtain more accurate statistic properties was performed by Whiting. In his study, instead of using quanta counting based monoenergetic x-ray source, polyenergetic x-ray beam was considered. The energy spectrum of x-ray quanta was converted to signal strength proportional to their energy by an analog-to-digital converter. The computed PDF obtained through Fourier transforming of the characteristic function and comparing to experimental PDF collected from phantom scans was found to follow the form of a compound Poisson process. The PDF would revert to the standard Poisson function or tend to a Gaussian function when certain criteria are satisfied. Studies on the moments of the detection process further validate the conclusion.

2.3 Simulation method

The CP model well describes the energy-integrating process in x-ray CT and thus is more close to the real physics. Based on this model, some researches have utilized well-known statistics functions, like Gamma, as the basic independent component in the CP model, to approximate the energy-integrating x-ray spectra and obtained promising results. However, since the actual photon spectrum is hard to be expressed by any known functions, the accuracy of this approximation needs further investigation.

Other than using well-known statistic functions to approximate the independent component in compound Poisson model, we use photon energy spectra from real CT system to seek for accurate estimation. To achieve this goal, we describe the compound Poisson model in x-ray CT as following:

Let X_i be the energy of the *i*th x-ray photon. It is a statistical random variable and its PDF is described by the energy spectrum of the x-ray source. Let N be the number of x-ray quanta detected by the detector during a time interval. It is also a statistical random variable and its PDF is described by a Poisson distribution with mean λ. Supposed that $\Phi(E)$ is the energy spectrum which is determined by the imposed voltage applied to the tube. Then the energy-integrating measurement recorded from the detector, Y, is a compound Poisson random variable and its PDF is described by the CP distribution. The mathematical model of this energy-integrating x-ray detection process is:

$$Y = \frac{\lambda^N e^{-\lambda}}{N!} \int_0^{Kev} \Phi(E) dE \tag{1}$$

In expression (1), it should be noted that though the real energy spectrum is continuous, it will be adapted to discrete form when it comes to practical implementation. Making use of the fact that the number of incident photon is statistically independent with the photon energy spectra, thus (1) could be further expressed as:

$$Y = \sum_{i=0}^{N} \Phi(E) \tag{2}$$

In expression (1), Y is a function of two random variables of X and N. According to the law of total expectation, the expectation of Y could be represented as:

$$E(Y) = E(E(Y \mid N)) \tag{3}$$

Since N and X are independent to each other, the expectation of Y could be further expressed as:

$$E(E(Y \mid N)) = E(NE(X)) = E(N)E(X) = \lambda E(X) \tag{4}$$

The variance of the compound Poisson random variable could be obtained through a similar derivation above, and the derivation could be expressed as:

$$\mathrm{var}(Y) = E(\mathrm{var}(Y \mid N)) + \mathrm{var}(E(Y \mid N)) = \lambda(\mathrm{var}(X) + E(X)^2) = \lambda E(X^2) \tag{5}$$

The first two statistical moments, mean and variance, of actual photon spectra, as shown in figure 1, could be calculated directly from (4) and (5). Since the energy spectrum of emitted photon varies only with the tube voltage applied voltage under certain experimental conditions, the two statistical moments could be considered as a valuable reference for further investigations.

2.4 Statistical approximation

Let μ and σ^2 be the mean value and variance obtained from x-ray data. We select Gaussian distribution, Poisson distribution, and Gamma distribution as the independent term in the compound Poisson model. Then, for each approximation approach, we have:

$$\lambda\mu_{gauss} = \mu \Rightarrow \mu_{gauss} = \mu / \lambda \tag{6}$$

$$\sigma^2 = \lambda(\sigma_{gau}^2 + \mu_{gau}^2) \Rightarrow \sigma_{gau}^2 = \sigma^2/\lambda - \mu^2/\lambda^2 \tag{7}$$

$$\mu = \lambda\sigma_{poi}^2 \Rightarrow \sigma_{poi}^2 = \mu/\lambda \tag{8}$$

$$\sigma^2 = \lambda(\sigma_{poi}^2 + \sigma_{poi}^2) \Rightarrow \sigma_{poi}^2 = \sigma^2 / 2\lambda \tag{9}$$

$$\mu = \lambda k\theta \Rightarrow k = \mu/\lambda\theta \tag{10}$$

$$\sigma^2 = \lambda(k\theta^2 + k^2\theta^2) \Rightarrow \theta = \sqrt{\sigma^2 / \lambda(k + k^2)} \tag{11}$$

where μ_{gau} and σ_{gau}^2 denote the mean value and variance of desired Gaussian distribution, μ_{poi} and σ_{poi}^2 are the mean and variance of desired Poisson distribution, and k and θ denote the scale and shape parameter of desired Gamma distribution, respectively. It should be noted that Eq. (10) and (11) could be further simplified by simultaneous equations.

3. RESULTS

To get the first two moments of the CP model, we estimated the mean and variance from the photon energy spectra obtained directly, and take used Eq. (4) and (5) to calculate them. The obtained mean and variance is the ground truth for the actual signal in this comparison study.

A comparison study was performed when a simple Poisson function was used to approximate the real spectra of Fig. 1. Because the mean and variance should always be the same for Poisson distribution, once the mean value has been set, the designated variance may not be achievable for Poisson substitute function in this study. Thus the error will be ineluctably introduced by using the Poisson function to approximate the actual spectra in the compound Poisson model of expression (1). We further studied the approximations by other substitute statistical functions, e.g., Gamma and Gaussian. Figure 2 show the PDF profiles to achieve the designated mean and variance when using Gamma and Gaussian functions. The errors can be significant. For example, at the 80 kVp scanning case, the error in variance estimation was 57.9% for the Poisson approximation. When the variances of Gamma and Gaussian functions were chosen to be 10%, 5% and 1% of the mean, respectively, the approximation by the substitute functions were 66.5%, 63.8% and 60.4%.

4. DISCUSSION AND CONCLUSION

In the field of x-ray computed tomography, Poisson model has been well studied in noise modeling and image reconstruction. This model, based on counting the number of emitted photon, explains general observed trends of signals with noise exposure and object attenuation. However, the ignoring of the energy-integrating detection will inevitably make the model differ from the real physics and thus will no longer be hold for seeking more accurate studies.

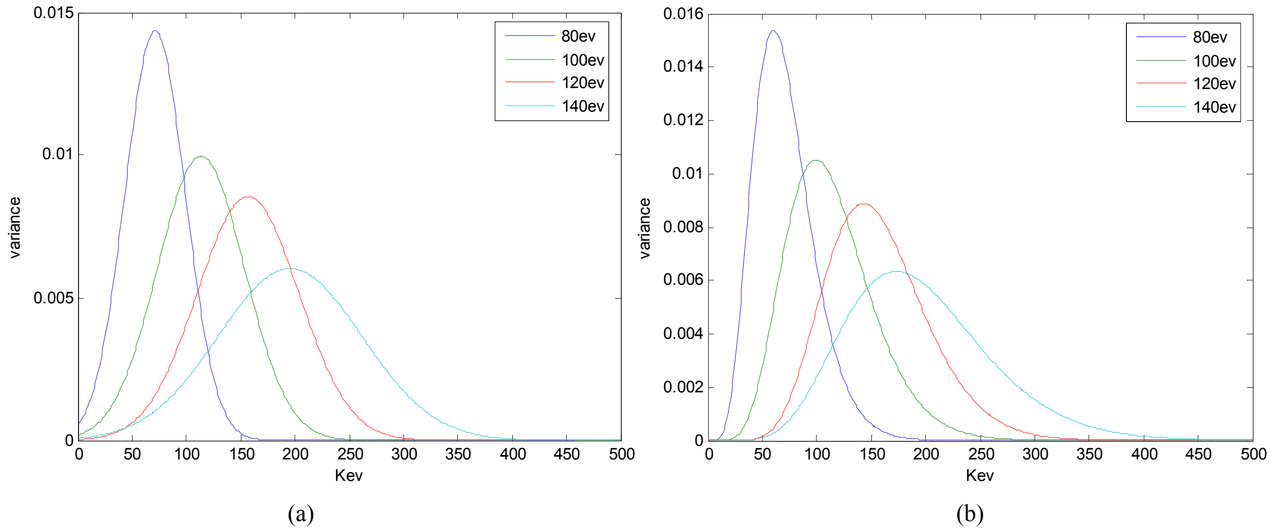

Fig. 2: Designated PDF profiles to approximate actual spectra: (a) Using Gamma substitute function. (b) Using Gaussian substitute function.

The methods and measurements presented in this paper are based on the compound Poisson model and allow a comparison between actual x-ray CT and the mathematic approximation. Based on the observation that the real x-ray CT data distribution consists of a continuous spectrum and spikes other than discrete values, we further investigate the compound Poisson model and compute the mean and variance of the measurement at any count-density level strictly. It provides a quantitative measure on any approximation to the compound Poisson statistics and also a statistical model to be used for adaptive noise treatment.

This work also includes a preliminary study which select commonly used statistic functions as independent term in the compound Poisson model to approximate the accrual photon spectra. To achieve an accurate approximation, we first use the x-ray spectra obtained from commercial X-ray systems and applied it to the compound Poisson model to determine the first two statistics moments. Then use the results as a desired output when applying the substitute functions. An estimation of the parameters for selected functions will be available based on the presented expression. The method used in this study may provide a good estimate on the first order statistics, i.e., the mean, but the error in estimating the second order statistics or variance can be significant.

Further efforts should be devoted to noise modeling and noise-reduction algorithm development based on the compound Poisson noise model (1), as well as the inclusion of electronic background noise is needed. The exact formula (4) for the second order statistics can be incorporated into a penalized weighted least-squares (PWLS) framework, which considers the second statistic as the weight during restoration, for noise treatment and image reconstruction.

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