Chapter 2

Image Formation

In this chapter, we describe mathematical models for the formation of projection data in both ECT (emission computed tomography) and TT (transmission tomography), for both PET and SPECT. The sensed projection data (sinogram) is the “image”, although as seen in Fig. 1.3, it does not look like a conventional image. The physics of image formation is described in a bit of detail for both PET and SPECT. Note that our goal is to estimate the “object” (the mean count rate at each image location during the scan) from the “image” (the projection data = number of counts at each detector location within a given duration).

2.1 Notational Conventions

Before we proceed, we describe the mathematical notation and conventions used in this thesis. In this thesis, unless otherwise stated, uppercase bold letters denote random fields or random vectors and corresponding uppercase italicized quantities represent random variables. The random variables carry an index denoting the component of the random field. Vectors are represented by lowercase bold symbols and corresponding vector elements by lowercase italicized symbols with an index. For instance, \( \Pr(G = g) \) is the probability that the discrete random vector \( G \) takes
the (vector) value \( \mathbf{g} \), and random variable \( G_i \) and scalar \( g_i \) are elements of \( \mathbf{G} \) and \( \mathbf{g} \) at location \( i \), respectively. If \( \mathbf{F} \) is a continuously-valued random field, then its probability density function (pdf) is denoted by \( P(\mathbf{F} = \mathbf{f}) \) or simply \( P(\mathbf{f}) \). The conditional probability, \( \Pr(\mathbf{G} = \mathbf{g}|\mathbf{f}) \), is the probability of \( \mathbf{G} \) taking the value \( \mathbf{g} \) given \( \mathbf{f} \), with a similar interpretation for conditional density \( P(\mathbf{G} = \mathbf{g}|\mathbf{f}) \).

As for matrices, they are denoted by calligraphic characters, and matrix elements are indicated by calligraphic characters with an index. For example, \( \mathbf{H}_{ij} \) is the \( ij \)th element of the matrix \( \mathbf{H} \). The transpose of a matrix \( \mathbf{H} \) and a vector \( \mathbf{f} \) is denoted by \( \mathbf{H}^T \) and \( \mathbf{f}^T \), respectively, and the inverse of the matrix \( \mathbf{H} \) by \( \mathbf{H}^{-1} \).

To avoid dealing with lots of summations, subscripts and superscripts, a component-operation notation is sometimes used as in \cite{155,12}. If \( \mathbf{a} \) and \( \mathbf{b} \) are two \( N \times 1 \) vectors, their component product is another \( N \times 1 \) vector \( \mathbf{d} \) such that \( \mathbf{d} = \mathbf{a} \mathbf{b} \) with \( \{d_j = a_j b_j; j = 1, ..., N\} \). The same idea can be used to calculate a ratio \( \mathbf{\frac{a}{b}} \) of two vectors as a component-by-component ratio \( \mathbf{\frac{a}{b}} = [\mathbf{\frac{a}{b}}]_j = \frac{a_j}{b_j} \), a logarithm of a vector log(\( \mathbf{a} \)) as the vector of logarithms of each component \( \log(a_j) = [\log(\mathbf{a})]_j \), and an exponential of a vector exp(\( \mathbf{a} \)) as the vector of exponentials of each component \( \exp(a_j) = [\exp(\mathbf{a})]_j \). However, a matrix-vector product \( \mathbf{H} \mathbf{a} \) is denoted in the usual way, and the conventional dot product will be distinguished from a component product by a transpose \( T \). Thus \( \mathbf{a}^T \mathbf{b} \) and \( \mathbf{a}^T \mathbf{H} \mathbf{b} \) are scalars and \( \mathbf{a} \mathbf{b} \) is a vector, and the \( i \)th component of vector \( \mathbf{H} \mathbf{a} \) is given by

\[
[\mathbf{H} \mathbf{a}]_i \equiv \sum_{j=1}^{N} \mathbf{H}_{ij} a_j \tag{2.1}
\]

A diagonal \( M \times M \) matrix with diagonal elements given by \( M \)-dimensional vector \( \mathbf{g} \) is denoted by \( \text{diag}(\mathbf{g}) \) with the \( i \)th diagonal element of the matrix equal to \( g_i \).
2.2 Image Formation - Basic Imaging Equations

In a linear digital imaging system, a continuous quantity \( f(x, y) \) is often sensed as a digital quantity \( g_i \) with \( i \) indicating the \( i \)th digital sensor of the system. For example, the digital image \( \mathbf{g} \) is recorded by a CCD (charge-coupled device) camera from the light intensity \( f(x, y) \) falling on the sensor. In general, a deterministic model for image formation of a 2-D digital imaging system can be described by the equation,

\[
g_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_i(x, y) f(x, y) \, dx \, dy
\]

(2.2)

where \( s_i(x, y) \) represents the contribution from object position \( (x, y) \) to the \( i \)th sensor measurement. For example, in a CCD camera, the amount of data in the \( i \)th pixel (sensor), \( g_i \), is the result of integrating the image light intensity \( f(x, y) \) over the area of the sensor at location \( i \). Note that we use a single index \( (i) \) so that elements \( \{g_i; i = 1, \ldots, M\} \) conveniently compose a vector \( \mathbf{g} \).

Emission computed tomography (ECT) is an example of a linear digital imaging system. In ECT, after administrating the radionuclide into a patient’s body, the radionuclide localizes in the tissue or interest, and then emits photons (gamma-rays) in all directions. Therefore, the object \( f(x, y) \) in ECT is the photon-emission rate, i.e. number of photons (on average) emitted into the \( 4\pi \) sphere per unit time, from the object at spatial location \( (x, y) \). (More detail about \( f(x, y) \) will be described shortly.) As for the digital sensor, in ECT, each detector element is usually called a detector “bin”, and it receives photons from the object. Thus, the observed data \( g_i \) is number of photons collected in the \( i \)th detector “bin”, where Eq.(2.2) holds.

Figure 2.1 illustrates a simplified tomographic imaging system. Here \( \mathbf{g} \) is a 1-D array with \( g_i \) denoting the physical detector bin \( i \) that collects the photons emitted from the 2-D object \( f(x, y) \). The support of the contribution function, \( s_i(x, y) \), is
shown as the shaded area in the 2-D object plane. Locations outside the shaded area have zero probability of contributing to bin $g_i$. Here, the shape of $s_i(x,y)$ is due to the presence of a collimator (short lines perpendicular to detector) that restricts acceptance of photons from all angles except to ones near $90^\circ$ to the detector face.

![Diagram of $s_i(x,y)$ and $g_i$]

Figure 2.1: A simplified tomographic imaging system is shown here. The dark area illustrates the contribution response from spatial location $(x,y)$ to bin $i$.

In reality, $s_i(x,y)$ is quite complex. It accounts for many physical effects such as scatter, attenuation, and detector response. More detail on the physical effects will be discussed later in this chapter. At this point, $s_i(x,y)$ is kept simply as a contribution function from spatial location $(x,y)$ to the $i$ th bin.

### 2.2.1 The Object Representation

For emission tomography, the primary signal for us to reconstruct is the 2-D radio-nuclide density, $f(x,y)$, which is a continuous function on a single cross-sectional plane. For simplicity and well-matched digital display, the continuous function is usually digitally sampled into an array of pixels, $\{f_j\}_{j=1,...,N}$, by integrating over the
\[ f_j = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) I_j(x, y) \, dx \, dy \]

where \( j = 1, \ldots, N \) and \( I_j(x, y) \) is an indicator function (i.e. a rect function =1 over area of pixel, =0 otherwise). Many other choices are possible for \( I_j(x, y) \). However, the rectangular pixels/voxels using the rect functions are the most popular one. Note that the pixel rect function is not a complete basis in that \( f(x, y) \) may not be recoverable. Therefore, we should replace the equal sign (=) with an approximation sign (\( \approx \)). We will use \( N \) to denote the number of pixels in the image. Thus, the 2-D continuous image \( f(x, y) \) has been transformed to a 1-D digital vector \( f \). In practice, the digitized image is displayed as a 2-D digital image matrix. One can get such a vector representation of a 2D discrete image matrix by, for example, by row-by-row scanning the matrix, and then concatenating one row after another [130]. Thus, \( f(x, y) \) can be approximated as a series expansion by \( f = \{f_j\}_{j=1, \ldots, N}, \)

\[ f(x, y) \approx \sum_{j=1}^{N} f_j I_j(x, y) \]  

(2.3)

### 2.2.2 The Projection Representation

By combining Eqs.(2.2) and (2.3), the projection data \( g_i \) become,

\[
g_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_i(x, y) f(x, y) \, dx \, dy
\]

\[
\approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_i(x, y) \sum_{j=1}^{N} f_j I_j(x, y) \, dx \, dy = \sum_{j=1}^{N} \mathcal{H}_{ij} f_j
\]

(2.4)

with

\[
\mathcal{H}_{ij} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_i(x, y) I_j(x, y) \, dx \, dy.
\]  

(2.5)
Thus, the relation between the \( i \) th bin, \( g_i \) and the digitized object \( f \) finally turns out to be

\[
g_i = \sum_{j=1}^{N} \mathcal{H}_{ij} f_j \tag{2.6}
\]

Here, \( \mathcal{H}_{ij} \) is related to \( s_i(x,y) \) via Eq.2.5 and \( i = 1,..,M \). Thus, \( \mathcal{H}_{ij} \) represents the contribution from pixel \( j \) to bin \( i \). (We have replaced the \( \approx \) in Eq.(2.4) by \( = \).)

In practice, \( g \) is recorded as a 2-D projection matrix by collecting the 1-D projection array shown in Fig. 2.1 bin by bin and angle by angle. Like the object \( f \), the 2-D projection matrix can be transformed into a 1-D vector in lexicographic ordering. Thus, \( g \) is a \( M \times 1 \) vector with \( g = \{ g_i, i = 1,..,M \} \), and \( M \) will be used to denote the total number of bins in the system. Note that \( f \) is a \( N \times 1 \) vector. Using our notational definitions we are entitled to write a shorthand version of Eq.(2.6)

\[
g = \mathcal{H} f \tag{2.7}
\]

where \( \mathcal{H} \) is an \( M \times N \) matrix.

The entire description has been done in terms of 2-D objects and images, but extension to 3-D follows readily simply by letting indices \( i \) and \( j \) run over 3 dimensions instead of 2. All the equations remain the same.

### 2.2.3 Poisson Noise Model

In nuclear medicine, the radioactive tracer atoms decay and then emit photons. It can be shown that [11, 155] the number of photons emitted from the tracer during a finite interval of time follows the Poisson distribution [49]. That is, the random variable \( F_j \) measuring the photons per unit time emitted into a \( 4\pi \) sphere from pixel \( j \) has a Poisson distribution with mean \( f_j \). From the statistical point of view, photons
emitted from pixel $j$ are either detected by the detectors or never recorded, and thus the detection follows a Bernoulli distribution [155]. Here, the Bernoulli probability is given by $H_{ij}$, the probability that photon emitted from the $j$th pixel gets detected in the $i$th detector bin. One can show that the cascaded random process of Poisson (with mean $f_j$) and Bernoulli (with probability $H_{ij}$) is Poisson [11, 155] with mean $H_{ij}f_j$ and that the counts received at different bins are independent. That is, the total number of photons emitted from source location $j$ and detected at bin $i$, denoted by the random variable $C_{ij}$, is the result of the cascaded Poisson and Bernoulli process, and is thus an independent Poisson distribution with mean $H_{ij}f_j$. The photon counts collected from all possible pixels at detector $i$, denoted by the random variable $G_i$, is equal to the summation $\Sigma_j C_{ij}$. Since the summation of many Poisson random variables is Poisson with mean equal to summation of all means, the detected photon counts $G_i$ follows a Poisson distribution with mean and variance equal to $\Sigma_j H_{ij}f_j$. Since the photon counts in different detector bins are independent [155], we can then write the joint probability of detected counts $G$ conditioned on the object $f$ as

$$
\Pr(G = \mathbf{g}|f) = \prod_{i=1}^{M} \Pr(G_i = g_i|f) = \prod_{i=1}^{M} \frac{e^{-\bar{g}_i} \bar{g}_i^{g_i}}{g_i!}
$$

(2.8)

where $\bar{g}_i = \Sigma_j H_{ij} f_j$ is the mean detected count rate at bin $i$.

Since a typical scan in nuclear medicine would consist relatively few detected photons (usually 500,000 photons total compared to $10^7$ X-ray photons/cm$^2$ in X-ray imaging or $10^{11}$-$10^{12}$ optical photons/cm$^2$ for a typical photographic film exposure [11]), it is not surprising that signal dependent Poisson noise is usually the dominant noise in nuclear medicine. This thesis will focus on the image reconstruction problem with
the statistical photon noise, the main source of uncertainty. Very similar imaging and estimation problems occur in other fields where photons are limited, such as gamma and X-ray astronomy \[54\].

### 2.2.4 Emission Projection Model

Since the projection data $g$ has Poisson distribution with mean $\bar{g}$,

$$G \sim \text{Poisson}(\bar{g}) = \text{Poisson}(\mathcal{H}f)$$

(2.9)

The projection “model” for the ECT can be written as a linear transformation,

$$G = \mathcal{H}f + N$$

(2.10)

where $N$ is the Poisson noise model. This model will be used for the statistical image reconstruction problems for ECT.

Here $\mathcal{H}$ is a system matrix with $M \times N$ elements, and is determined by the system model. Theoretically, $\mathcal{H}_{ij}$ can be interpreted as the probability that a photon emitted from pixel $j$ will be detected by detector bin $i$. Also $\mathcal{H}_{ij}$ can be determined physically by simply putting a point source at square pixel $j$ and measuring the ratio of counts at bin $i$ to total counts emitted. All the elements can be decided in this manner. In practice, $\mathcal{H}_{ij}$ is calculated by some approximation to a real physical model, or precomputed once and then stored as a matrix in computer memory. Since most of the elements in $\mathcal{H}$ are zeros, $\mathcal{H}$ is actually very sparse.

Here, in this thesis, we often compute $\mathcal{H}_{ij}$ by a simple model using a single-ray approximation described in [138]. As indicated in Fig. 2.2, $\mathcal{H}_{ij}$ is represented by $l_{ij}$, the length of intersection of the $i$ th projection ray and the $j$ th pixel, $\mathcal{H}_{ij} = l_{ij}$. It turns out that $\mathcal{H}_{ij}$ needn’t be equal to the probability of detecting a photon in bin $i$ from pixel $j$, but simply proportional to it. Normalization will be easily incorporated.
Figure 2.2: The system matrix \( H_{ij} \) is calculated as a simple chord length \( l_{ij} \).

Since \( H_{ij} \) is proportional to the probability of detection of photons in bin \( i \) from pixel \( j \), longer \( l_{ij} \) means larger detection sensitivity. However, for more accurate modeling, physical effects such as scatter, attenuation, detector response, and random coincidences (PET), which degrade the system sensitivity, must be accounted for in the system model [150, 109], and will be discussed in next section.

### 2.3 Degrading Effects on the Image - Basic Physical Models

For the range of photon energies used in nuclear medicine, the important physical factors degrading quantitation are detector response, attenuation, and scatter both in SPECT and PET, and random coincidences only in PET. These factors are listed in table 2.1. In order to provide more accurate image reconstructions that contain reduced image artifacts and distortions, all the degrading effects from various factors must be modeled in the reconstruction process [150]. Here, we will qualitatively
describe all the factors which reduce the image quality in SPECT and PET, and in the next section discuss quantitative models.

<table>
<thead>
<tr>
<th>Image Degradation Factors</th>
<th>SPECT</th>
<th>PET</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometrical Detector Response</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Attenuation</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Scatter</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Randoms</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Detector Efficiency</td>
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<td>✓</td>
</tr>
<tr>
<td>Dead-Time</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Positron Range and Non-Collinearity</td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 2.1: Physical factors degrading quantitation in SPECT and PET.

### 2.3.1 Geometrical Detector Response

The geometrical detector response is proportional to the area of detector seen by the photons emitted from the object. It is equal to the probability that a photon emitted at \( j \) in air will hit the detector bin \( i \). This is purely a geometrical effect. In SPECT, because of the use of a collimator, the geometric response is characterized by the spatial design of the collimator-detector system [150], and thus by the acceptance angle of the collimator holes. For a parallel hole collimator in SPECT shown in Fig. 2.3, the geometric point spread function broadens as the distance from the collimator increases. Therefore, the resolution distance decreases with decreasing distance from source to detector. For PET, the geometrical response should consider the solid angle subtended at each of the detectors by each pixel [58, 82].

### 2.3.2 Photon Attenuation

As photons travel through body tissues, they are attenuated due to the photo-electric effect and Compton scattering [150, 141]. The photo-electric effect is an atomic absorption process in which the energy of an incident photon is absorbed
Figure 2.3: Geometrical detector response is illustrated in a parallel-hole collimator-detector SPECT system. Photons originating at A have more spread than those from C.

totally by the atom and an electron is ejected. Compton scattering is a collision between a photon and a free electron. The photon loses part of its energy and changes direction after the scattering process, and some of the photons are scattered out of the field of view and are never detected. Therefore, those photons absorbed totally by the tissue, and those scattered out of the field of view are the contributions to the attenuation.

Attenuation, usually measured by a linear attenuation coefficient with units of cm$^{-1}$, measures the reduction of the number of detected gamma rays by photo-electric absorption and Compton scatter [141]. If a narrow “pencil” beam of photons of intensity $I_o$ is transmitted through a medium of constant attenuation coefficient $\mu$, the intensity of the beam drops according to Beer’s Law [141],

$$ I(x) = I_o e^{-\mu x} $$  \hspace{1cm} (2.11)

where $I(x)$ is the photon intensity after passing a distance $x$ through the medium. The attenuation coefficient $\mu$ depends on a number of factors including photon energy, scattering cross-section of the material, and electron density [141, 150]. As shown in Fig. 1.5, D indicates an absorbed photon, B is a scattered photon and the photon at C is flying out of field of view.
The attenuation effect is extremely important, because it causes significant data loss and resulting image errors. The attenuation coefficient has typical values for water of 0.15 cm⁻¹ for 140 KeV photons (typical SPECT energy) and 0.096 cm⁻¹ for 511 KeV photons (PET energy) [8]. Due to the fact that human body is composed primarily of water, the effects of attenuation for photons emitted from inside the body can be quite significant. For example, only 20% to 25% of photons survive to reach the detector for typical brain and cardiac SPECT studies [150] due to the attenuation effect.

Figure 2.4 illustrates one reconstruction artifact due to the attenuation. Fig. 2.4(a) is a uniform circle emission phantom with a corresponding uniform attenuation coefficient equal to 0.15 cm⁻¹, and (b) shows a reconstructed image of the circle phantom but without attenuation correction. Since Fig. 2.4(a) and (b) are displayed in the same gray scale, the artifact of low contrast can be seen in (b) compared to (a). Also a profile plot along the center row of these two images is shown in Fig. 2.4(c). Note that the large amount of intensity reduction and "cupping" artifact due to attenuation is best illustrated in the profile plot.

![Figure 2.4](image.png)

Figure 2.4: This illustrates the effect of attenuation on the reconstructed image. (a) A uniform circle emission phantom with attenuation coefficient equal to the value (0.15 cm⁻¹) of water at 140 KeV, (b) the reconstructed image without attenuation correction, and (c) the profile plot of (a) (solid line) and (b) (dashed line) along the center row.
The attenuation map is usually not uniform, but varies with position in a complicated way. For example, in the thorax, with very different attenuation coefficients of the lung, muscle and bone tissues, the attenuation effect is much more complicated. One needs to have patient specific information on the spatial distribution of attenuation coefficients in order to calculate and model for the attenuation. Transmission tomography is concerned with getting the attenuation map, and the model for it will be discussed later.

2.3.3 Scatter

The attenuation effect is associated with a photon that is absorbed or scattered and not detected by any detector. Some attenuated photons are not absorbed, and are scattered into a new direction such that they do get detected by a detector.

For a scattered photon, its energy is related to the scattering angle $\theta$ under the law of energy and momentum conservation, and given by [141],

$$ E_s = \frac{E_o}{1 + \frac{E_o}{\Delta E}(1 - \cos \theta)} $$

(2.12)

where $E_o$ and $E_s$ are the energies (in MeV) of the incident and scattered photons, respectively. Therefore, after the Compton scattering, a photon will loss some of its energy, and change its direction. The larger the scattering angle, the greater the loss of energy from the incident photon.

Scatter depends on source location, the geometry and composition of the medium, energy of the photon, energy resolution of the detector, and energy window used in imaging [74]. Since scattered photons have less energy than the unscattered photons, one can thus separate them using energy discrimination, i.e rejecting photons that are below the energy of the primary photons. However, because of the finite
energy resolution of the detector, a certain number of scattered photons are still
detected [150]. In PET, the energy loss for most forward scattered events is small
even for significant angular deviations of the photon [18], and this, combined with
poor energy resolution for PET detectors, makes the energy thresholding difficult.

Since there is a change of direction, if the scattered photon is detected and not
rejected, it will be recorded in a location different from where it would have hit. For
SPECT, the scatter is illustrated in Fig. 1.5 in chapter 1, where photon B experiences
a scatter and is mis-recorded as coming from an incorrect location. The scatter event
in PET is shown in Fig. 2.5, where the photon going in direction OB changes its
direction to C due to the single scatter, and is treated as if it were coming from
LOR AC. Note that, only a single scatter is encountered in Fig. 2.5; however it is
possible for both photons to scatter at the same time or experience scattering more
than once [105].

Usually, the fraction of detected scattered counts in a typical cardiac and brain
SPECT study is on the order of 20% to 40% of the total counts [150], while for PET,
the detected scattered events from a 20cm diameter phantom of water is 30% of the
primary events [18]. The scatter effect can lead to a significant background which will
reduce the intensity contrast of a reconstructed image, and impose a resolution blur
particularly at the edges between high activity and lower activity regions [18]. Scatter
affects the quality and quantitative accuracy for both PET and SPECT, and thus
needs to be modeled correctly in order to obtain an accurate image reconstruction.
However, the scatter response is complex, and modeling scatter is complicated and
difficult [105].
2.3.4 Randoms or Accidental Coincidences in PET

In PET, when only one of two photons at an annihilation is detected in its LOR due to scatter or small geometric solid angle, the event is called \textit{single}. As shown in Fig. 2.6, if two singles arising from two different annihilations are recorded at a pair of detectors within the same timing window, it is called a \textit{random} or \textit{accidental coincidences}. The randoms rate $R_i$ at the $i$th detector pair is related to the singles rate and given by

$$R_i = 2\tau S_{i1} S_{i2}$$

where $\tau$ is the coincidence timing window width, while $S_{i1}$ and $S_{i2}$ refer to the singles rates of the $i$th detector pair, and $i1$ and $i2$ are the two detectors corresponding to the $i$th detector pair. Usually 30\% or more of the detected events are accidental coincidences [120]. Randoms are a primary source of background noise and image
Figure 2.6: Diagram shows a random coincidence event in PET. Two singles (OC and EA) generated by two separate coincident events are detected within the same timing window (AC), and this accidental coincidence is called random coincidence.

distortion in PET. Thus, the effect of not modeling randoms in the image formation process is to add a roughly uniform background activity to the reconstructed image [57, 101].

2.3.5 Other Effects

Detector Efficiency: The detector efficiency is the probability that photons get recorded when entering the crystal, and it depends on the density and thickness of the scintillation crystal. Thus it is a pure detector effect. The detector efficiency can be measured and corrected through system calibration procedures.

Dead-time: The dead-time or pulse resolving time refers to the time required to process individual detected events. While a detected event is being processed, the detector is not able to process another. Therefore, if a second signal pulse occurs before the first has disappeared, it does not generate a detected signal. That is, the
effect of dead-time causes the loss of detected photons, thus reduced sensitivity. The fraction of counts loss due to dead-time is proportional to the true count rate at low to moderate counts rate, but is non-linear at high counts rate [150, 109]. For most SPECT studies, the average count rate is relatively low, resulting in a dead-time loss of only a few percent [150], thus it is usually ignored. For PET, the dead-time correction factor is often estimated from the singles rate [82].

**Positron Range and Non-collinearity in PET:** There are two fundamental processes which limit the spatial resolution in PET [101, 141]. The first effect comes from the finite range of positrons before annihilation. The magnitude of the positron range depends on the positron energy, and the density of the surrounding tissue, and varies widely among different isotopes. For example, the FWHM of positron range for $^{18}$F is 0.1mm, and for $^{15}$O is 0.5 mm. Since it is not known in which direction the positron drifts, this effect is equivalent to a blur uncertainty of diameter equal to the positron range. The second degrading factor on resolution is due to the divergence from collinearity of two annihilation photons produced by a coincident event. The degree of non-collinearity depends on the momentum of the positron and election when they annihilated, and is on the order of one degree or less [109]. Two photons leaving at 181° for example, would hit a different detector pair than if they left at 180°. This is equivalent to blurring the LOR’s due to the angular uncertainty. Compared to the intrinsic resolution of most PET scanners, these are not serious source of errors, and are usually ignored [109].

### 2.4 Emission Model for SPECT and PET

The physical effects have been qualitatively described above. Here, we would like to mathematically model these physical effects for both SPECT and PET. We
Consider the emission model first. The transmission model will be mentioned next section.

As noted in section 2.2, the emission projection model has a Poisson distribution with mean equal to $\tilde{g}$, and

$$\tilde{g} = Hf + r$$

(2.14)

The emission model can be illustrated in Fig. 2.7 where $f_j$ is the mean emission rate at pixel $j$. Since most of the physical effects are usually linear, they can be captured in the system matrix $H$. Linear effects include geometrical response, detector efficiency, and attenuation. The background events $r$ are those physical effects which can be modeled as an additive term. It is also the mean of a Poisson distribution, but is denoted simply by $r$. This includes the randoms in PET. As for scatter, it is somewhat complicated, and will be explained later.
2.4.1 Geometrical Detector Response Model

Without any other physical effects, the geometrical response is the probability that photons emitted from a source location gets recorded at a given detector bin. As described in previous section, the geometrical response \( \mathcal{H}_{geo} \) can be determined physically by simply putting a point source in air at square pixel \( j \) and measuring the response at bin \( i \). As for SPECT, the geometrical response can often be modeled by a convolution kernel with a depth-independent [72] or depth-dependent [163] blurring process. However, in practice, \( \mathcal{H}_{geo} \) is often represented by some approximation to a real physical model. In our work \( \mathcal{H}_{ij}^{geo} \) is precomputed as the length \( l_{ij} \) of the \( i \)th ray passing through the \( j \)th pixel, and then stored as a matrix in computer memory. This simple line model is used in this thesis.

2.4.2 Attenuation Model

In a medium of thickness \( x \) with attenuation coefficient \( \mu \), the amount of transmitted photons \( N(x) \) is given by Beer’s law [11] as in Eq. (2.11),

\[
N(x) = N_0 e^{-\mu x}
\]

where \( N_0 \) is the unattenuated number of photons incident to the medium. If the attenuator is made up of a few materials of various compositions, then the product \( \mu x \) inside the exponent is replaced by a sum of the individual attenuation coefficient times the thickness of each material,

\[
N(x) = N_0 e^{-\int \mu(x) \, dx}
\]

The attenuation experienced in both SPECT and PET is a little different due to the nature of photon emission. For SPECT, as indicated in Fig. 2.8, the attenuation
experienced by a photon emitted inside the body is equal to $e^{-\int_{l_a} \mu(x) \, dx}$ with a non-
uniform monochromatic attenuation coefficient $\mu$. The magnitude of attenuation is 
dependent on the path length $l_a$ from the source to the detector, and thus a function 
of photon location in SPECT. That is, photons that are generated deeper inside 
the body experience more attenuation than those close to the surface [150]. Such 
an effect can be seen in an uncompensated, reconstructed image with a generally 
lower count density in the central region of the image as illustrated in Fig. 2.4. 
Since the attenuation is depth dependent, attenuation correction for SPECT becomes 
complicated.

The attenuation effect for SPECT can be modeled as $H_{\text{atten}}$ with $ij$th element 
equal to

$$H_{ij}^{\text{atten}} = \exp(- \sum_{k \in N_{ij}} \mu_k l_{ik}) \tag{2.15}$$

where $N_{ij}$ is the set of pixels along the $i$th ray from the $j$th pixel to the detector, for 
$i = 1, \ldots, M; j = 1, \ldots, N$, and $\mu_k$ the attenuation coefficient at pixel $k$.

For PET, since it involves a pair of annihilation photons, the attenuation encountered 
by a detected coincidence in even an inhomogeneous attenuator is equal to the product 
of those experienced by two oppositely directed 511KeV photons,

$$e^{-\int_{l_a} \mu(x) \, dx} e^{-\int_{l_b} \mu(x) \, dx} = e^{-\int_{a+b} \mu(x) \, dx} = e^{-\int_{\text{LOR}} \mu(x) \, dx}$$

where $\text{LOR}$ is the total path length along the line of annihilation response [141]. 
The attenuation factor for PET is thus independent of the origin of the annihilation 
along the line of response (LOR) [8]. Thus, considering the geometrical response and 
attribution, for a distributed positron source along any projection, the mean detected
Figure 2.8: Attenuation length \( l_a \) experienced by a SPECT photon is dependent on the position of the emitting pixel along the projection ray, while for PET, the attenuation length is \( l_a + l_b = D \). Thus, the attenuation for the projection in PET is fixed for a given line of response, and is dependent of the source location along the ray.

count \( \tilde{g}_i \) in Eq. (2.4) at any bin \( i \) for PET can be written as,

\[
\tilde{g}_i = \sum_j H_{ij} f_j
\]

\[
= \sum_j H_{ij}^{geo} \exp(-\sum_{k \in \mathcal{N}_i} \mu_k l_{ik}) f_j
\]

\[
= \exp(-\sum_{k \in \mathcal{N}_i} \mu_k l_{ik}) \sum_j H_{ij}^{geo} f_j
\]

where \( \mathcal{N}_i \) is the set of pixels along the LOR of projection \( i \), and \( H_{ij}^{geo} \) is the geometrical response. Since the attenuation in the LOR is independent of position along the LOR, the term, \( \exp(-\sum_{k \in \mathcal{N}_i} \mu_k l_{ik}) \) can be then taken out of the summation. (This is not the case for SPECT since photons are recorded only in one direction.) Therefore, the attenuation effect for PET can be modeled as \( H_{atten} \) with \( i \)th row equal to

\[
H_{atten}^i = \exp(-\sum_{k \in \mathcal{N}_i} \mu_k l_{ik})
\]
where \( i = 1, \ldots, M \).

Note that the inverse term, \( \exp(\sum_{k \in N_i} \mu_k l_k) \), is often called the attenuation correction factor (ACF) in PET, and can be simply obtained using a PET transmission measurement,

\[
ACF = \frac{N_o}{N(x)}
\]

where \( N(x) \) is the transmission count rate with object in the scanner, and \( N_o \) the blank scan without object in the scanner. Attenuation correction for PET can be accomplished by just multiplying the emission projection data by the ACF. This works well at high transmission count levels but fails when counts are low due to statistical fluctuations.

### 2.4.3 Scatter

Scatter can be modeled either as a linear effect or an additive term which is distributed as Poisson and has its own mean. The linear model is more accurate but less practical than the affine modeling. In SPECT, scatter is often modeled linearly as a convolution of the uncorrected projection data with an average scatter response function [39, 40, 157, 162], and thus can be also modeled in the system matrix. However, this model is usually calculated on line by a computer program [84], and difficult to express as a matrix. Other than modeling scatter in the system matrix, scatter can be modeled as affine term \( r \). To get \( r \), one can use a variety of techniques based on energy discrimination [64, 73, 106, 1] or single scatter calculation based on the Klein-Nishina formula [105, 108]. Since both scattered and unscattered counts are Poisson distributed, the scatter component can be modeled as an additive term to the primary counts.
2.4.4 Randoms

As described above, the randoms can be estimated by a delaying timing window. That is, two separate timing windows with a fixed amount of delay between them are used. The number of coincidences in the delayed window (photon $\tau$1 in timing window $\tau$1 and photon $\tau$2 in timing $\tau$2) can be used to estimate the contribution of counts due to randoms only. Therefore, a separate randoms sinogram is collected (acquired simultaneously with the standard emission sinogram) [105]. Note that usually the randoms are precollected in the machine by subtracting the randoms from the projection data containing the true and randoms coincidences. Nevertheless, the data follows a Poisson distribution, as do the randoms. Therefore, such precorrection increases variance due to the subtraction of two Poisson processes, and makes the result non-Poisson [82, 120]. That is why an additive term $r$ (the mean of a Poisson distribution) for randoms is preferred in the model [82] in order to reflect the nature of Poisson noise.

The other way to get the randoms component $r$ is to estimate the singles rate and use the Eq.(2.13) to compute it. For example, [105] described a procedure to estimate the randoms.

2.4.5 Other Physical Effects

Other effects such as detector efficiency, positron range and non-collinearity in PET can be modeled in the system matrix $\mathcal{H}$. For both SPECT and PET, the detector efficiency matrix is usually provided from the camera system through calibration procedures [82]. The effect of positron range and non-collinearity in PET can be modeled as a local blurring convolution [143, 82], and this can be obtained through a Monte Carlo simulations [103]. The amount of dead-time effect is usually
estimated from the singles rate [82]. However, the effects excluding detector efficiency
are of second order in comparison to geometrical response, attenuation, scatter, and
randoms, and thus usually are ignored.

All the physical effects can thus be factored into the system matrix as

\[ H = H^\text{geo} H^\text{atten} H^\text{others} \]  \hspace{1cm} (2.19)

for SPECT with the geometrical response \( H^\text{geo} \), the attenuation \( H^\text{atten} \), and other
effects \( H^\text{others} \) such as detector efficiency, and dead-time. As for scatter, it can
be modeled implicitly in the system matrix but calculated through a procedure
implemented in the computer program, or as an affine term \( r \) in the model.

The system matrix

\[ H = H^\text{geo} H^\text{atten} H^\text{others} \]  \hspace{1cm} (2.20)

for PET, with the geometrical response \( H^\text{geo} \), the attenuation \( H^\text{atten} \), and other effects
\( H^\text{others} \) accounting for detector efficiency, positron range, non-linearity and dead-
time [82]. Also the randoms and scatter in PET can be modeled as an additive term
\( r \) which is itself the mean of a Poisson process. Therefore, we can use the Poisson
noise model Eq.(2.8) with the mean given by Eq.(2.14) with the system matrix \( H \)
and the background events by \( r \), for emission tomography. We shall use this model
for the ECT reconstruction problem in the next chapter.

2.5 Transmission Model

Here we turn attention to the modeling of transmission tomography. As mentioned
in previous chapter, the signal in transmission tomography for us to obtain or reconstruct
is the attenuation map \( \mu \), which is required in the ECT reconstruction in order
to compensate the attenuation effect. As shown in Fig. 1.9 of chapter 1, for a transmission camera, whether in SPECT or PET, a radioactive source is located outside a patient’s body, and the detectors are on the opposite side of the body. Thus, we can use Fig. 2.9 to represent the diagram of the transmission projection model for both SPECT and PET. Note that compared to Fig. 2.7, now the attenuation coefficient $\mu_j$ at pixel $j$ takes the role of the object in TT instead of mean emission rate $f_j$ as in ECT.

The external source $\mathbf{U}$ is itself a random vector having a Poisson distribution with mean equal to $\hat{u}$. Since the count rate is so high for the external source, we in essence know $\hat{u}$ from measurements, and we denote this by $\mathbf{u}$. The external source emits photons that pass through the body to reach the detectors. When passing through the body, the photons are either recorded by the detector or never recorded, and thus the detection also follows a Bernoulli distribution. If we consider only the attenuation effect, the Bernoulli probability is the attenuation that photons experience when emitted from the source to the detector. That is, by Beer’s law [141] (Eq. (2.11)),

![Diagram](image-url)

Figure 2.9: The diagram for transmission tomography.
we have the attenuation equal to the exponentiated line integrals of the attenuation map \( \mu \):

\[
\exp(- \sum_{j \in \mathcal{N}_i} l_{ij} \mu_j) \equiv \exp(- \sum_{j \in \mathcal{N}_i} \mathcal{H}_{ij} \mu_j) \tag{2.21}
\]

where \( \mathcal{H}_{ij} \equiv l_{ij} \) is the chord length of ray \( i \) passing through pixel \( j \), and \( \mathcal{N}_i \) is the set of pixels along the \( i \)th projection ray. Note that the attenuation map \( \mu \) is unknown, and we have used \( \mathcal{H} \) indicating the system matrix, in this case, the chord length.

The detected counts \( \mathbf{G} \) are then the result of the cascaded Poisson-Bernoulli process, and follows Poisson distribution as in ECT. How about other physical effects? The physical effects described for ECT also apply to TT. Scatter (SPECT, PET) and randoms (PET) are usually modeled as an additive term \( \mathbf{r} \) as in ECT [104, 35]. The crosstalk contaminations [8, 97] from ECT when simultaneous acquisitions of ECT and TT are performed can be modeled in \( \mathbf{r} \) as well [35]. Other physical effects are usually of second order in error compared to scatter and randoms, and thus are ignored.

The transmission projection model has a Poisson distribution as in the emission case in Eq.(2.8) but with a different mean,

\[
\tilde{g}_i = u_i e^{-\sum_{j \in \mathcal{N}_i} \mathcal{H}_{ij} \mu_j} + r_i \tag{2.22}
\]

and \( i = 1, \ldots, M \). Data for blank scan source strength \( u_i \) are collected before data for \( g_i \), and to reduce statistical variation in \( \mathbf{u} \), the blank scan is usually acquired for a longer duration than a normal transmission scan.
Figure 2.10: These two images illustrate (a) a physically acquired blank scan $u$ without an object in the system, and (b) a physically acquired transmission projection data $g$ with a thorax phantom (Data Spectrum, Chapel Hill, NC). Both are acquired from a GE Advance PET scanner equipped with a rotating rod transmission sources. In both cases, the abscissa indicates the angles, and ordinate indexes the detector bins. Note that the darker area in the center part of (b) shows the reduced counts due to the attenuation effects inside the body.

2.6 Summary

In this chapter, we have described the projection formation of both emission and transmission tomography, and the nature of the system matrix $\mathcal{H}$ as well. Projection data in both ECT and TT follow Poisson distributions, each with their own definition of mean. A quick summary of the chapter is contained in the following equations:

$$g \sim \text{Poisson}(\tilde{g}) \quad \text{ECT and TT}$$
$$\tilde{g} = \mathcal{H}f + r \quad \text{ECT}$$
$$\tilde{g} = uf^{-\mu} + r \quad \text{TT}$$

For ECT, the unknown is the emission object $f$, and for TT, the unknown is the attenuation map $\mu$. Note we have used $g$ to indicate projection data for both ECT and TT, but the context will always be clear. In the next chapter, we shall summarize the image reconstruction methods both in ECT and TT using the models discussed above.